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MODELS OF PEANO ARITHMETIC AND A QUESTION OF SIKORSKI ON ORDERED FIELDS

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ABSTRACT

Using models of Peano Arithmetic, we solve a problem of Sikorski by showing that the existence of an ordered field of cardinality λ with the Bolzano–Weierstrass property for κ-sequences is equivalent to the existence of a κ-tree with exactly λ branches and with no κ-Aronszajn subtrees.

In a series of several papers [18], [19], [20] Sikorski considered the general problem of extending various algebraic and topological properties of the reals which depend, usually implicitly, on the cardinal parameter $\aleph_0$ to properties depending on some uncountable regular cardinal. For example, an ordered field has the Bolzano–Weierstrass property if every bounded sequence has a convergent subsequence. (By a sequence we mean, of course, an $\aleph_0$-sequence.) More generally, for a regular cardinal κ, an ordered field F is said to have the Bolzano–Weierstrass property for κ-sequences (briefly, F is BW(κ)) if $|F| \geqq \kappa$ and every bounded κ-sequence has a convergent κ-subsequence. Sikorski [18] constructs for each uncountable regular κ a BW(κ) ordered field of cardinality κ. (See Corollary 2.7 for an essentially different construction of a BW(κ) ordered field.) He mentions as an open problem (p. 88 of [18]) the existence of a BW(κ) ordered field of cardinality $>\kappa$. Our main result in this paper is the following theorem.

THEOREM. *Suppose $\lambda \geqq \kappa > \aleph_0$ are cardinals with κ regular. Then the following are equivalent:*

(1) *There is a* BW(κ) *ordered field of cardinality λ.*

(2) *There is a κ-tree with exactly λ branches and with no κ-Aronszajn subtrees.*

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There had already been very strong evidence for this theorem. It was shown by Cowles and LaGrange [3], using a theorem of Juhász and Weiss [7], that the existence of a BW(κ) ordered field of cardinality $>\kappa$ implies the existence of a κ-Kurepa tree with no κ-Aronszajn subtrees. Jensen (see Devlin [4]) had shown the existence of an $\aleph_1$-Kurepa tree with no $\aleph_1$-Aronszajn subtrees, assuming $V = L$. (The correctness of this proof has been questioned; at best, it is rather involved.) Subsequently, Devlin [6] gave a simpler construction of such trees, also assuming $V = L$. It is known from results of Solovay and Silver [21] that the nonexistence of an $\aleph_1$-Kurepa tree is equiconsistent with the existence of an inaccessible cardinal. However, the exact consistency strength of the nonexistence of an $\aleph_1$-Kurepa tree with no $\aleph_1$-Aronszajn subtrees is unknown. Todorčević [22] showed that there is a model in which, for every regular cardinal κ, there is a κ-Kurepa tree with no κ-Aronszajn subtrees. Then Devlin [5], and later Velleman [23], proved that the constructible universe L is such a model.

The first example of a BW(κ) ordered field of cardinality $>\kappa$ was given by Manevitz and Miller [12] who showed that it is consistent to assume the existence of such fields when $\kappa = \aleph_1$. They mention that Velleman subsequently proved, assuming $V = L$, that for every regular κ there is a BW(κ) field of cardinality κ^+.

The proof of the hard direction (2) $\Rightarrow$ (1) of the Theorem makes use of a construction of models of Peano Arithmetic built along a tree. Such models were first constructed in [13], and this construction was corrected and refined in [14]. That these models are useful in constructing ordered fields was first observed by Keisler in [9], which is an unpublished manuscript summarized by [10].

1. Ordered fields and trees

In this section we discuss some preliminaries concerning ordered fields and trees and the relationships between them.

Scott [16] defined the notion of a complete ordered field, and later Keisler [9] referred to such ordered fields as Scott complete. An initial segment I of an ordered field F is *Dedekindean* if for each positive $\delta \in F$ there is $x \in I$ such that $x + \delta \notin I$. Then F is *Scott complete* if every Dedekindean initial segment has a supremum in F.

The following basic result is proved in [16].

THEOREM 1.1. *Let F be an ordered field. Then there is a Scott complete ordered field $\hat{F}$ in which F is dense. Furthermore, $\hat{F}$ is unique up to isomorphism over F.*

The ordered field $\hat{F}$ in the theorem is the *Scott completion* of F. Note that the

Scott completion need not be real-closed; however, if F is real-closed then so is $\hat{F}$.

Because of the presence of the linear order on an ordered field F, it makes sense to refer to its cofinality $\mathrm{cf}(F)$ (which is called the character of F in [18]). This is an important cardinal invariant of F because of the following observation of Sikorski. If κ is a regular cardinal and F has a convergent κ-sequence which is not eventually constant, then $\mathrm{cf}(F)=\kappa$.

A κ-sequence $\langle a_\nu : \nu < \kappa\rangle$ is *Cauchy* if for every positive $\delta \in F$ there is some $\nu < \kappa$ such that $|a_\alpha - a_\beta| < \delta$ whenever $\nu < \alpha, \beta < \kappa$. Cowles and LaGrange [3] defined an ordered field F to be *κ-complete* if $\mathrm{cf}(F)=\kappa$ and every Cauchy κ-sequence converges. One easily deduces that F is κ-complete iff F is Scott complete and $\mathrm{cf}(F)=\kappa$.

According to Keisler [9] a subset N of an ordered field F is an *integer set* if the following three conditions are met: (1) N is a set of non-negative elements of F; (2) $0 \in N$; and (3) for each positive $x \in F$ there is a unique $n \in N$ such that $n \leqq x < n+1$. Notice that every ordered field has an integer set. The integer set is unique only for Archimedean ordered fields; however, any two integer sets of an ordered field are order-isomorphic.

PROPOSITION 1.2. *Every ordered field F has an integer set which is closed under addition.*

PROOF. For the purposes of this proof let us call an additive subgroup $G \subseteq F$ *discrete* if 1 is the only element $x \in G$ such that $0 < x \leqq 1$. A discrete subgroup G is *closed* iff whenever $0 < k < \omega$ and $y \in F$, where $ky \in G$ and $G \cup \{y\}$ generates a discrete subgroup, then $y \in G$. The desired conclusion follows from the following three facts.

(1) There is a discrete subgroup.

(2) Every discrete subgroup is a subgroup of a closed discrete subgroup.

(3) If G is a closed discrete subgroup and $a \in F$ is such that for no $b \in G$, $a \leqq b \leqq a+1$, then $G \cup \{a\}$ generates a discrete subgroup.

To obtain the desired conclusion, let G be a maximal discrete subgroup, the existence of which follows easily from (1) and Zorn's Lemma. By (2) G is closed, so it follows from (3) that the set $N = \{x \in G : x \geqq 0\}$ is an integer set closed under addition.

We now prove (1)–(3). Fact (1) is trivial, and by Zorn's Lemma so is (2). We assume (3) is false and derive a contradiction.

Suppose $G \cup \{a\}$ does not generate a discrete subgroup. Then there is a least positive $k < \omega$ such that for some $x \in G$

$$(*) \qquad 0 < ka - x < 1.$$

Let $y = x/k$. Clearly then, $y \notin G$ because $a < y + 1 < a + 1$. Since G is closed, this implies that $G \cup \{y\}$ generates a subgroup which is not discrete. Therefore, there are $z \in G$ and positive $n < k$ such that

$$(**) \qquad 0 < ny - z < 1.$$

Multiplying (*) by n/k and adding this to (**) yields

$$0 < na - z < n/k + 1.$$

Thus, either

$$0 < na - z < 1$$

or

$$0 < na - (z + 1) < 1.$$

But either case contradicts the minimality of k. ∎

Cowles [2] generalizes the notion of an Archimedean ordered field to a κ-Archimedean ordered field, where κ is any infinite cardinal. An ordered field is *κ-Archimedean* iff every integer set is κ-like. Clearly, the Scott completion of a κ-Archimedean ordered field is κ-Archimedean. In fact, any integer set for F is also an integer set for $\hat{F}$. (Keisler [9] defines when an ordered field is *monotone complete*, and then proves that if $\mathrm{cf}(F) = \kappa$, then F is monotone complete iff F is Scott complete and κ-Archimedean.)

Cowles and LaGrange [3] make the following definition. An ordered field F is *κ-Ramsey* iff $\mathrm{cf}(F) = \kappa$ and every κ-sequence has a monotone κ-subsequence. They then prove the following characterization of $\mathrm{BW}(\kappa)$.

THEOREM 1.3. *An ordered field F is* $\mathrm{BW}(\kappa)$ *iff it is κ-Archimedean, κ-Ramsey and Scott complete.*

We next turn to trees. A *tree* is a partially ordered set $(T, <)$ such that for any $a \in T$, the set $\hat{a} = \{x \in T : x < a\}$ of its predecessors is well-ordered. The order type of $\hat{a}$ is the *rank* of a, denoted by $\mathrm{rk}(a)$. The *height* of a tree T is $\mathrm{ht}(T) = \{\mathrm{rk}(a) : a \in T\}$. For each $\nu < \mathrm{ht}(T)$ we let $T_\nu = \{a \in T : \mathrm{rk}(a) = \nu\}$. A *branch* B of T is a linearly ordered subset of T such that $B \cap T_\nu \neq \varnothing$ for each $\nu < \mathrm{ht}(T)$. We let $[T]$ denote the set of branches of T.

For a regular cardinal κ, we say that T is a *κ-tree* if $\mathrm{ht}(T) = \kappa$ and $|T_\nu| < \kappa$ for each $\nu < \kappa$.

A κ-tree T is a *κ-Aronszajn* tree provided $[T]=\emptyset$; it is a *κ-Kurepa* tree provided $|[T]|>\kappa$.

Keisler [9] associated trees with each non-Archimedean ordered field F in the following way. Suppose that $\mathrm{cf}(F)=\kappa$ and that $\langle c_\nu : \nu<\kappa\rangle$ is an increasing, cofinal sequence of positive elements of F such that $c_{\nu+1}/c_\nu$ is infinitely large for each $\nu<\kappa$. For each $\nu<\kappa$ let $\sim_\nu$ be an equivalence relation defined on $[0,1]^F$, the closed unit interval of F, according to the following: $x\sim_\nu y$ iff $|x-y|c_\nu$ is not infinitely large. Let T be the set of all equivalence classes of all the equivalence relations $\sim_\nu$. Then $(T,\supseteq)$ is a tree. We will refer to such a tree as a Keisler tree of F.

PROPOSITION 1.4. *Let F be a non-Archimedean ordered field, $\kappa=\mathrm{cf}(F)$, and T a Keisler tree of F. Then $\mathrm{ht}(T)=\kappa$, and for each $x\in[0,1]^F$ there is a unique $B\in[T]$ such that $\bigcap B=\{x\}$. Furthermore:*

(1) *F is Scott complete iff $\bigcap B\neq\emptyset$ for each $B\in[T]$;*

(2) *F is κ-Archimedean iff T is a κ-tree;*

(3) *F is $\mathrm{BW}(\kappa)$ iff T is a κ-tree with no κ-Aronszajn subtrees.*

The proof, except for (3), is at least implicit in Keisler [9]. Statement (3) follows easily from Theorem 1.3 and the following lemma, which we leave unproved, but which is easily proved in the style of the proof of Proposition 2.4 of [22].

Let $(T,<)$ be a tree, and suppose $\prec$ is some linear order of T. The ordering causes $[T]$ to be ordered lexicographically. We will refer to such linear orders on $[T]$ as lexicographical orders without specifying $\prec$.

LEMMA 1.5. *Let $(T,<)$ be a κ-tree for some regular κ such that $T=\bigcup[T]$ and $[T]$ is ordered lexicographically. Then the following are equivalent:*

(1) *T does not have a κ-Aronszajn subtree.*

(2) *Every subset of $[T]$ of cardinality κ contains a subset of cardinality κ which is either well-ordered or inversely well-ordered.*

COROLLARY 1.6. (1) *If there is a κ-Archimedean, Scott complete ordered field of cardinality λ, then there is a κ-tree with exactly λ branches.*

(2) *If there is a $\mathrm{BW}(\kappa)$ ordered field of cardinality λ, then there is a κ-tree with exactly λ branches but with no κ-Aronszajn subtrees.*

Corollary 1.6(1) is from Keisler [9]; Cowles and LaGrange [3] give another argument for successor κ. Corollary 1.6(2) is essentially proved in [3] by first showing that a κ-Archimedean ordered field F is $\mathrm{BW}(\kappa)$ iff $[0,1]^F$ is κ-compact, and then invoking the Juhász–Weiss result [4] that there is a κ-metrizable, κ-compact space of cardinality λ iff there is a κ-tree with exactly λ branches and with no κ-Aronszajn subtrees.

Keisler [9] also proved the converse to a less precise version of Corollary 1.6(1) by showing that if there is a κ-Kurepa tree with exactly λ branches, then there is a κ-Archimedean, Scott complete ordered field of cardinality *at least* λ. We will prove in Corollaries 3.4 and 3.3 the exact converses of both parts of Corollary 1.6.

2. Models of Peano Arithmetic

This section is concerned with models of Peano Arithmetic (PA), especially in regard to their connection with the Theorem stated in the introduction.

The construction of the reals from the natural numbers is a very familiar one, and can be done in various ways which result in the same ordered field. One standard way is first to construct the ordered field of quotients, and then to let the reals be the completion of this field. This same sort of construction works for arbitrary models $\mathcal{N}$ of Peano Arithmetic: first form the ordered field of quotients, and then its Scott completion. We will denote this resulting ordered field by $\mathbf{R}^{\mathcal{N}}$ and refer to its elements as *reals* of $\mathcal{N}$. Clearly, the set N is an integer set of $\mathbf{R}^{\mathcal{N}}$.

If $\mathcal{N}$ is a model of PA, then a subset $X \subseteq N$ is a *class* if for any $a \in N$, the set $\{x \in X : x \leqq a\}$ is definable in $\mathcal{N}$. Let Class($\mathcal{N}$) be the set of classes of $\mathcal{N}$. We will describe $\mathbf{R}^{\mathcal{N}}$ in another way showing the close relationship between the reals of $\mathcal{N}$ and the classes of $\mathcal{N}$. Correspond with each class X the formal infinite series $\Sigma_{i\in X} 2^{-(i+1)}$. For each $a \in N$ let

$$s_a = \sum_{\substack{i\in X \\ i\leqq a}} 2^{-(i+1)}.$$

Since X is a class, s_a is in the field of quotients and hence is a real of $\mathcal{N}$. It is easily seen that the initial segment

$$\{x \in \mathbf{R}^{\mathcal{N}} : x < s_a \text{ for some } a \in N\}$$

is Dedekindean, and therefore has a supremum in the closed unit interval of $\mathbf{R}^{\mathcal{N}}$. We will denote this supremum by $r(X)$. For every real x of $\mathcal{N}$ in the closed unit interval there is a class X (which is unique except for the usual redundancy occurring in decimal expansions) so that $r(X) = x$. Specifically, r is a bijection from the set of all classes X for which $N \setminus X$ is unbounded to the half-open unit interval $[0,1)$ of $\mathbf{R}^{\mathcal{N}}$.

We now easily obtain the following proposition observed by Keisler [6].

PROPOSITION 2.1. *For any model $\mathcal{N}$ of PA*, $|\mathbf{R}^{\mathcal{N}}| = |\mathrm{Class}(\mathcal{N})|$.

It is also quite easy to see that the next proposition is true.

PROPOSITION 2.2. *For any model $\mathcal{N}$ of PA*, $\mathbf{R}^{\mathcal{N}}$ *is real-closed.*

To prove Proposition 2.2, consider some polynomial $p(x)$ over the quotient field of $\mathcal{N}$ such that for elements q_0, q_1 in the quotient field, $0 < q_0 < q_1 < 1$ and $p(q_0) < 0 < p(q_1)$. Then one easily constructs an $X \in \mathrm{Class}(\mathcal{N})$ such that $q_0 < r(X) < q_1$ and $p(r(X)) = 0$.

Each model $\mathcal{N}$ of PA has associated with it a partially ordered set $(N, \lhd)$ which fails to be a tree only because it is not well-founded (unless $\mathcal{N}$ is the standard model). To be definitive we make the following definitions in PA. Let $\mathrm{lh}(x) = k$ iff $2^k \leqq x + 1 < 2^{k+1}$. For each $d < \mathrm{lh}(x)$, let $(x)_d \in \{0, 1\}$ be the d-th digit in the binary expansion of $x + 1$. Finally, let $x \lhd y$ iff $\mathrm{lh}(x) < \mathrm{lh}(y)$ and for each $d < \mathrm{lh}(x)$, $(x)_d = (y)_d$. For a model $\mathcal{N}$ of PA, a subset $B \subseteq N$ is a *branch* if B is linearly ordered by $\lhd$ and for each $k \in N$ there is $x \in B$ such that $\mathrm{lh}(x) = k$. Each branch of $\mathcal{N}$ is a class. There is a natural one-to-one correspondence between branches and classes in which the branch B is associated with the class $\{d \in N : (x)_d = 1$ for some $x \in B$ such that $d < \mathrm{lh}(x)\}$.

We say that a subset $X \subseteq N$ is a *cover* in $\mathcal{N}$ if $X \cap B \neq \varnothing$ for each branch B of $\mathcal{N}$. Then, $\mathcal{N}$ is *compact* if each cover has a bounded subcover. That the standard model of PA is compact is essentially just König's Lemma.

PROPOSITION 2.3. *If $\mathcal{N}$ is compact, then $\mathcal{N}$ is κ-like for some regular κ.*

PROOF. Suppose $\mathcal{N}$ is not κ-like for any regular κ. Let $\lambda = \mathrm{cf}(\mathcal{N})$. Then there is a sequence $\langle b_\alpha : \alpha < \lambda \rangle$ of distinct elements of N, where $\mathrm{lh}(b_\alpha) = \mathrm{lh}(b_0)$ for each $\alpha < \lambda$. Let $\langle c_\alpha : \alpha < \lambda \rangle$ be an unbounded sequence of elements of N such that $b_\alpha \lhd c_\alpha$ for each $\alpha < \lambda$. Let $C_0 = \{c_\alpha : \alpha < \lambda\}$ and let $C = \{x \in N : C_0 \cup \{x\}$ is an antichain$\}$.

Clearly $C_0 \subseteq C$ and C is a cover in $\mathcal{N}$. Any subcover of C must include C_0 and so must be unbounded. Therefore, $\mathcal{N}$ is not compact. ∎

PROPOSITION 2.4. *Let $\mathcal{N}$ be κ-like for some regular κ. Then $\mathcal{N}$ is compact iff $(N, \lhd)$ has no κ-Aronszajn subtrees.*

PROOF. Let $A \subseteq N$ be a κ-Aronszajn tree, and let $C = \{x \in N :$ for no $a \in A$ is $x \leqq a\}$. Clearly, C is a cover, for if $B \cap C = \varnothing$ for some branch B, then $B \cap A \in [A]$. It is also clear that C has no bounded subcover. Let $C_0 \subseteq C$ be a subcover, and let $a \in A$ be such that $\mathrm{lh}(a)$ is arbitrarily large. Let $B \subseteq N$ be a branch such that $a \in B$. Then, since C_0 is a subcover, there is $x \in C_0 \cap B$. Clearly $a \lhd x$ so that $\mathrm{lh}(x) > \mathrm{lh}(a)$. Thus C_0 is unbounded. This proves $\mathcal{N}$ is not compact.

Conversely, suppose $\mathcal{N}$ is not compact, and let $R \subseteq N$ be well-ordered and unbounded. Let $C = \{c_\nu : \nu < \kappa\}$ be a cover with no bounded subcover. We can assume that if $c_\mu \lhd c_\nu$, then $\nu < \mu$. Let $X \subseteq C$ be a maximal antichain. Then X is unbounded; for, if $\mathrm{lh}(x) < a$ for each $x \in X$, then $\{c \in C : \mathrm{lh}(c) < a\}$ would be a bounded subcover of C. Then $A = \{a \in N : a \lhd x$ for some $x \in X$, and $\mathrm{lh}(a) \in R\}$ is a κ-Aronszajn subtree of $(N, \lhd)$. ∎

The next proposition gives the reason for introducing compact models of PA.

PROPOSITION 2.5. *Let $\mathcal{N}$ be a model of PA. Then $\mathbf{R}^{\mathcal{N}}$ is* BW(κ) *iff $\mathcal{N}$ is compact and* $\mathrm{cf}(\mathcal{N}) = \kappa$.

PROOF. Suppose $\mathbf{R}^{\mathcal{N}}$ is BW(κ). Then $\mathcal{N}$ is κ-like since it is an integer set, and $\mathbf{R}^{\mathcal{N}}$ is κ-Ramsey by Theorem 1.3. Then, from Lemma 1.5 it follows that $(N, \lhd)$ has no κ-Aronszajn subtrees, so by Proposition 2.4, $\mathcal{N}$ is compact.

Conversely, suppose $\mathcal{N}$ is compact and $\mathrm{cf}(\mathcal{N}) = \kappa$. By Proposition 2.3, $\mathcal{N}$ is κ-like. Again using Lemma 1.5, we can get that $\mathbf{R}^{\mathcal{N}}$ is κ-Ramsey, hence BW(κ). ∎

Our aim, therefore, becomes to construct compact models of Peano Arithmetic having many classes. As a warm-up for the next section, we will construct some compact models. The natural way to get such models produces models which have an additional interesting property. Recall from [8] or [15] that a model $\mathcal{N}$ of PA is *rather classless* if each class of $\mathcal{N}$ is definable. Rather classless models are most easily constructed by means of chains of conservative extensions. If $\mathcal{N} \prec \mathcal{M}$, then $\mathcal{M}$ is a *conservative* extension of $\mathcal{N}$ if $X \cap N$ is definable in $\mathcal{N}$ whenever X is definable in $\mathcal{M}$. Conservative extensions of models of PA are necessarily end extensions. The extension $\mathcal{N} \prec \mathcal{M}$ is *simple* if there is some $a \in M \setminus N$ such that $\mathcal{M}$ has no proper, elementary substructure containing $N \cup \{a\}$. The fundamental theorem of MacDowell–Specker [11] says that every model of PA has a simple, conservative extension.

PROPOSITION 2.6. *Let $\mathcal{N}_0$ be a model of PA and $\kappa > |N_0|$ a regular cardinal. Then $\mathcal{N}_0$ has a conservative extension $\mathcal{N}$ which is compact, κ-like and rather classless.*

PROOF. Form a continuous chain $\langle \mathcal{N}_\nu : \nu < \kappa \rangle$ of models such that each $\mathcal{N}_{\nu+1}$ is a simple, conservative extension of $\mathcal{N}_\nu$. Let $\mathcal{N} = \bigcup \{\mathcal{N}_\nu : \nu < \kappa\}$. Clearly, $\mathcal{N}$ is a

κ-like, conservative extension of $\mathcal{N}_0$. It is rather classless by Lemma 3.1 of [15]. Now suppose that $(N, \lhd)$ has a κ-Aronszajn subtree A. Let $H = \{x \in N : x \unlhd a \text{ for some } a \in A\}$. Then, H contains no branches of $\mathcal{N}$. Since κ is regular and uncountable, there is $\alpha < \kappa$ such that $(\mathcal{N}_\alpha, H \cap N_\alpha) < (\mathcal{N}_\alpha, H)$. Let $b \in H \setminus N_\alpha$ and then set $B_\alpha = \{x \in N_\alpha : x \lhd b\}$. Since all the extensions are conservative, B_α is a definable branch of $\mathcal{N}_\alpha$. The same formula defining B_α in $\mathcal{N}_\alpha$ defines a branch B of $\mathcal{N}$. But $B \subseteq H$ since $B_\alpha \subseteq H$, and this is a contradiction. ∎

COROLLARY 2.7. *For each consistent extension Σ of PA and each regular $\kappa > \aleph_0$, there is a* BW(κ) *ordered field with an integer set which is a model of Σ.*

We remark that the model $\mathcal{N}$ constructed in the proof of Proposition 2.6 has the property that for each regular $\lambda > |N_0|$, $(N, \lhd)$ has no Aronszajn λ-subtree.

Based on the proof of Proposition 2.6 one might think that for regular κ, every κ-like, rather classless model of PA is compact. That is, however, not the case. In the next section we give a definitive result (Corollary 3.5) in this regard. For now, we present another example.

PROPOSITION 2.8. *Suppose $\mathcal{N}$ is a recursively saturated, rather classless model of PA. Then $\mathcal{N}$ is not compact.*

PROOF. By Proposition 2.3 we can assume that $\mathcal{N}$ is κ-like for some regular κ. The proof of Theorem 3 of [15] shows that $(N, \lhd)$ has a κ-Aronszajn subtree. By Proposition 2.4, $\mathcal{N}$ is not compact. ∎

It is shown in [15] that for possibly many (or even all) uncountable regular cardinals κ there is a κ-like, recursively saturated, rather classless model of PA. For $\kappa = \aleph_1$, Kaufmann [8] and Shelah [17] had already proved such models exist absolutely.

COROLLARY 2.9. *There is an $\aleph_1$-like, rather classless, recursively saturated model of PA which is not compact.*

3. The construction

We begin this section by summarizing the results of [13] and [14].

Let $\mathcal{N}$ be a model of PA and $n < \omega$. A subset $D \subseteq N^n$ is *dense* if whenever $a_0, a_1, \ldots, a_{n-1} \in N$, then there is $\langle d_0, d_1, \ldots, d_{n-1}\rangle \in D$ such that $a_1 \lhd d_i$ for each $i < n$. Let I be some index set such that B_i is a branch of $\mathcal{N}$ for each $i \in I$. Then $\langle B_i : i \in I\rangle$ is an *indexed family of mutually generic branches* of $\mathcal{N}$ iff whenever $n < \omega$, $i_0, i_1, \ldots, i_{n-1}$ are distinct and $D \subseteq N^n$ is a definable dense set, then $(B_{i_0} \times B_{i_1} \times \cdots \times B_{i_{n-1}}) \cap D \neq \varnothing$.

In [14] a recursive set $\Phi = \{\phi_k(x) : k < \omega\}$ of formulas in the language of PA is defined. This set Φ is used to construct extensions of models of PA in the following manner. Let $\langle B_i : i \in I \rangle$ be an indexed family of mutually generic branches of $\mathcal{N}$. For each $i \in I$ let b_i be a new individual constant, and consider the theory

$$\Sigma_I = \{\phi_k(b_i) : i \in I, k < \omega\} \cup \{\mathrm{lh}(b_i) = \mathrm{lh}(b_j) : i, j \in I\}$$
$$\cup \{a \lhd b_i : a \in B_i\} \cup \mathrm{Th}((\mathcal{N}, a)_{a \in N}).$$

This theory is complete and has a minimal model $\mathcal{M}$, which is an end extension of $\mathcal{N}$. For any $J \subseteq I$ there is a unique model $\mathcal{M}_J$, where $\mathcal{N} \precneqq \mathcal{M}_J < \mathcal{M}$, such that $\{b_j : j \in J\} = M_J \cap \{b_i : i \in I\}$. Each $\mathcal{M}_J$ is cofinal in $\mathcal{M}$ (so $\mathrm{cf}(\mathcal{M}) = \aleph_0$). Furthermore, if $X \in \mathrm{Class}(\mathcal{M}_J)$, then $X \cap N$ is definable in $(\mathcal{N}, \langle B_j : j \in J \rangle)$.

We will say that $\mathcal{M}$ is the canonical extension of $\mathcal{N}$ by $\langle B_i : i \in I \rangle$.

Now let us consider a model $\mathcal{N}$ and a tree $(T, <)$ of height α. Some mild normality conditions need to be required of T. We will impose upon T more than is needed. Let us say that T is *normal* if for each $\nu < \alpha$ the following hold:

(1) if $\nu = 0$, then $|T_\nu| = 1$;

(2) if ν is a limit ordinal and $t \in T_\nu$, then $|\{s \in T_\nu : \hat{s} = \hat{t}\}| = 1$ (where $\hat{s} = \{r \mid r < s\}$);

(3) if $\nu = \mu + 1$ and $t \in T_\nu$, then $|\{s \in T_\nu : \hat{s} = \hat{t}\}| \leqq 2$.

We will now build an extension of $\mathcal{N}$ along T under the assumption that T is normal. This is accomplished by constructing an elementary chain $\langle \mathcal{N}_\nu : \nu < \alpha \rangle$, and then setting $\mathcal{M} = \bigcup\{\mathcal{N}_\nu : \nu < \alpha\}$. Associated with each $\mathcal{N}_\nu$ is an indexed family $\langle B_t : t \in T_\nu \rangle$ of mutually generic branches.

Let $\mathcal{N}_0$ be some simple, conservation extension of $\mathcal{N}$, so that $\mathrm{cf}(\mathcal{N}_0) = \aleph_0$. Let $\langle B_t : t \in T_0 \rangle$ be an indexed family of mutually generic branches.

Suppose we have $\mathcal{N}_\nu$ with indexed family $\langle B_t : t \in T_\nu \rangle$ of mutually generic branches. Let $\mathcal{N}_{\nu+1}$ be the canonical extension of $\mathcal{N}_\nu$ by $\langle B_t : t \in T_\nu \rangle$. Since $\mathrm{cf}(\mathcal{N}_{\nu+1}) = \aleph_0$, there is an indexed family $\langle B_t : t \in T_{\nu+1} \rangle$ of mutually generic branches such that if $s \in T_\nu$, $t \in T_{\nu+1}$ and $s < t$, then $b_s \in B_t$.

Next suppose that ν is a limit ordinal, and that for each $\mu < \nu$ we have constructed $\mathcal{N}_\mu$ together with its associated indexed family $\langle B_s : s \in T_\mu \rangle$. Let $\mathcal{N}_\nu = \bigcup\{\mathcal{N}_\mu : \mu < \nu\}$. For each $t \in T_\nu$, let $B_t = \bigcup\{B_s : s < t\}$. Then $\langle B_t : t \in T_\nu \rangle$ is an indexed family of mutually generic branches of $\mathcal{N}_t$.

Finally, let $\mathcal{M} = \bigcup\{\mathcal{N}_\nu : \nu < \alpha\}$. For each branch B of T, let $X_B = \bigcup\{B_t : t \in B\}$. Then $\langle X_B : B \in [T] \rangle$ is an indexed family of mutually generic branches of $\mathcal{M}$. In particular, if $\mathcal{M}$ is built along T, then $|\mathrm{Class}(\mathcal{M})| \geqq |[T]|$. We refine this inequality in the following theorem.

THEOREM 3.1. *Suppose* $(T,<)$ *is a normal tree such that* $\mathrm{ht}(T)=\alpha$, *where* $\mathrm{cf}(\alpha)\geqq\aleph_1$, *and suppose* $\mathcal{M}$ *is built along* T. *Then every class of* $\mathcal{M}$ *is definable in* $(\mathcal{M},\langle X_B : B\in[T]\rangle)$.

PROOF. Let $X\in\mathrm{Class}(\mathcal{M})$; we wish to show that X is definable in $(\mathcal{M},\langle X_B : B\in[T]\rangle)$.

For each $\nu<\alpha$, $X\cap N_\nu$ is definable in $(\mathcal{N}_\nu,\langle B_t : t\in T_\nu\rangle)$. Therefore, there is some finite $I\subseteq T_\nu$ such that $X\cap N_\nu$ is definable in $(\mathcal{N}_\nu,\langle B_t : t\in I\rangle)$. Let I_ν be the intersection of all such finite I. We claim that $X\cap N_\nu$ is definable in $(\mathcal{N}_\nu,\langle B_t : t\in I_\nu\rangle)$. To verify this claim, it suffices to show that if $I,I'\subseteq T_\nu$ are finite and $X\cap N_\nu$ is definable in both $(\mathcal{N}_\nu,\langle B_t : t\in I\rangle)$ and $(\mathcal{N}_\nu,\langle B_t : t\in I'\rangle)$, then $X\cap N_\nu$ is definable in $(\mathcal{N}_\nu,\langle B_t : t\in I\cap I'\rangle)$. We will show this in a special, but typical, case. Suppose $I=\{s\}$ and $I'=\{t\}$, with $s\neq t$, and $X\cap N_\nu$ definable in both $(\mathcal{N}_\nu,B_s)$ and $(\mathcal{N}_\nu,B_t)$. Let $\phi(B_s,x)$ and $\psi(B_t,x)$ be the corresponding defining formulas. Using the genericity of $\langle B_s,B_t\rangle$, there are conditions $\rho_s\in B_s$ and $\rho_t\in B_t$ such that $\langle\rho_s,\rho_t\rangle$ forces $\forall x(\phi(B_s,x)\leftrightarrow\psi(B_t,x))$. But then ρ_s must decide every instance of $\phi(B_s,x)$, so that $a\in X\cap N_\nu$ iff ρ_s forces $\phi(B_s,a)$.

By Fodor's Theorem, there are $\beta<\alpha$, $n<\omega$, a formula $\phi(x,y,X_0,X_1,\dots,X_{n-1})$, and an unbounded $A\subseteq\alpha$ which have the following properties whenever $\nu\in A$. First of all, $|I_\nu|=n$, so let $I_\nu=\{t_0,t_1,\dots,t_{n-1}\}$ be arranged so that $B_{t_0},\dots,B_{t_{n-1}}$ are in lexicographical order. Set $B_i^\nu=B_{t_i}$. Then, whenever, $i<j<n$, then $B_i^\nu\cap N_\beta\neq B_j^\nu\cap N_\beta$. Finally, there is some $a\in N_\nu$ such that $\phi(x,a,B_0^\nu,B_1^\nu,\dots,B_{n-1}^\nu)$ defines $X\cap N_\nu$ in $(\mathcal{N}_\nu,B_0^\nu,B_1^\nu,\dots,B_{n-1}^\nu)$.

Let a_ν be the least such a in the previous paragraph. Notice that if $\mu<\nu$ are both in A, then $X\cap N_\mu$ is definable in $(\mathcal{N}_\mu,B_0^\nu\cap N_\mu,\ B_1^\nu\cap N_\mu,\dots,B_{n-1}^\nu\cap N_\mu)$. Thus, in fact, $B_i^\mu=B_i^\nu\cap N_\mu$ for $i<n$. We now claim that if $\mu<\nu$ are both in A, then $a_\mu=a_\nu$. Clearly, $a_\mu\leqq a_\nu$, since $(\mathcal{N}_\alpha,B_0^\mu,\dots,B_{n-1}^\mu)<(\mathcal{N}_\nu,B_0^\nu,\dots,B_{n-1}^\nu)$. In the structure $(\mathcal{N}_\mu,B_0^\mu,\dots,B_{n-1}^\mu)$, the formulas $\phi(x,a_\nu,B_0^\mu,\dots,B_{n-1}^\mu)$ and $\phi(x,a_\mu,B_0^\mu,\dots,B_{n-1}^\mu)$ both define the same set (namely $X\cap N_\mu$), so they both define the same set (namely $X\cap N_\nu$) in $(\mathcal{N}_\nu,B_0^\nu,\dots,B_{n-1}^\nu)$. But then $a_\nu\leqq a_\mu$. Therefore, $a_\mu=a_\nu$.

Let $a=a_\nu$ for $\nu\in A$. Let $B_i=\bigcup\{B_i^\nu : \nu\in A\}$. Then clearly $\phi(x,a,B_0,\dots,B_{n-1})$ defines X in $(\mathcal{N},B_0,B_1,\dots,B_{n-1})$, completing the proof. ∎

THEOREM 3.2. *Let* $(T,<)$ *be a normal* κ*-tree, for regular* κ, *with no* κ*-Aronszajn subtrees, and let* $\mathcal{N}$ *be a model of PA with* $|N|<\kappa$. *Suppose* $\mathcal{M}$ *is an extension of* $\mathcal{N}$ *built along* T. *Then* $\mathcal{M}$ *is compact.*

PROOF. Clearly $\mathcal{M}$ is κ-like, so by Proposition 2.4 we need only show that

$(M, \lhd)$ has no κ-Aronszajn subtrees. To the contrary, suppose A is a κ-Aronszajn subtree of $(M, \lhd)$. Let $\{c_\nu : \nu < \kappa\}$ be an antichain such that $c_\nu \not\in \{x \in M : x \leqq c$ for some $c \in A \cup \{c_\mu : \mu < \nu\}\}$.

We can now use Fodor's Theorem to get some $\gamma < \kappa$, a term $\tau(x_0, x_1, \ldots, x_n)$ in the language of PA allowing constants from N_γ, and an unbounded $S \subseteq \kappa$ such that for each $\nu \in S$ there are $t_{\nu,0}, t_{\nu,1}, \ldots, t_{\nu,n} \in T_\nu$ such that $\tau(b_{t_{\nu,0}}, b_{t_{\nu,1}}, \ldots, b_{t_{\nu,n}}) = c_\nu$. Furthermore, since T is a κ-tree with no κ-Aronszajn subtrees, we can assume there are branches $B_0, B_1, \ldots, B_n \in [T]$ such that for each $i \leqq n$ and $t \in B_i$, $b_i \lhd b_{t_{\nu,i}}$ for sufficiently large $\nu \in S$. Thus, for each $\mu < \kappa$, all $(n+1)$-tuples $\langle b_{t_{\nu,0}}, b_{t_{\nu,1}}, \ldots, b_{t_{\nu,n}} \rangle$, for sufficiently large $\nu \in S$, realize the same type over $\mathcal{N}_\mu$. Thus, all c_ν, for sufficiently large $\nu \in S$, realize the same type over $\mathcal{N}_\mu$. Thus, we get for each $\mu < \kappa$ a unique d_μ such that $\mathrm{lh}(d_\mu) = \mathrm{lh}(c_\mu)$ and $d_\mu \lhd c_\nu$ for all sufficiently large $\nu \in S$. Each $d_\mu \in A$, and if $\mu < \nu < \kappa$, then $d_\mu \lhd d_\nu$. Thus, $\{d_\mu : \mu < \kappa\}$ contradicts A being a κ-Aronszajn subtree of $(M, \lhd)$. ■

The following corollary to the two previous theorems is an improvement of the Theorem stated in the Introduction.

COROLLARY 3.3. *Suppose $\lambda \geqq \kappa > \aleph_0$ are cardinals with κ regular. Suppose Σ is some consistent extension of PA. Then the following are equivalent*:

(1) *There is a* BW(κ) *ordered field of cardinality λ.*

(2) *There is a* BW(κ) *real-closed field of cardinality λ having an integer set which is a model of Σ.*

(3) *There is a κ-tree with exactly λ branches and with no κ-Aronszajn subtrees.*

PROOF. This is immediate from Corollary 1.6(2), Theorems 3.1 and 3.2, and the observation that the existence of a tree as in (3) is equivalent to the existence of a normal such tree. ■

Similarly, we can obtain the following result, which slightly improves Keisler [9].

COROLLARY 3.4. *Suppose $\lambda \geqq \kappa > \aleph_0$ are cardinals with κ regular. Suppose Σ is some consistent extension of PA. Then the following are equivalent*:

(1) *There is a Scott complete, κ-Archimedean ordered field of cardinality λ.*

(2) *There is a Scott complete, κ-Archimedean real-closed field of cardinality λ having an integer set which is a model of Σ.*

(3) *There is a κ-tree with exactly λ branches.*

The equivalence of (3) and (4) in the next corollary was promised at the end of §2.

COROLLARY 3.5. *Suppose* $\kappa > \aleph_0$ *is regular and* Σ *is a consistent extension of PA. Then the following are equivalent*:

(1) *There is a* κ*-Archimedean ordered field which is not* BW(κ).

(2) *There is a* κ*-Archimedean real-closed field which is not* BW(κ) *having an integer set which is a model of* Σ.

(3) *There is a* κ*-Aronszajn tree.*

(4) *There is a* κ*-like, rather classless model of* Σ *which is not compact.*

Thus, we see that if there is a κ-like model of PA which is not compact, then there is one which is also rather classless.

4. Addendum

The techniques used to prove the theorems of the previous sections can also be used to prove analogous results in other settings. In this section we will briefly discuss p-valued fields. (Cherlin [1] is a good reference for this section.)

We will consider fields K which are equipped with a valuation $v : K \to G \cup \{\infty\}$, where G is an ordered abelian group. A subset $X \subseteq K$ is *bounded* iff there is some $b \in G$ such that $v(x) \geqq b$ for each $x \in X$. The basic open sets for the topology on K are those of the form $\{x \in K : v(x - x_0) \geqq b\}$, where $x_0 \in K$ and $b \in G$. For a regular cardinal κ we make the following definition: a valued field K is BW(κ) iff $|K| \geqq \kappa$ and every bounded κ-sequence has a convergent κ-subsequence.

We fix for the remainder of this section a prime number p. A valued field K is *p-valued* if (1) F has characteristic 0, (2) the residue field is F_p, the prime field of characteristic p, and (3) $v(p)$ is the least positive element of G. In addition, if K satisfies Hensel's Lemma and G is a **Z**-group (that is, G is elementarily equivalent to $(\mathbf{Z}, +, <)$, the ordered group of integers), then K is *p-adically closed.* The archetypical p-adically closed field is $\boldsymbol{Q}_p$, the field of p-adic numbers.

THEOREM 4.1 (Ax–Kochen, Ershov). *A valued field K is p-adically closed iff* $K \equiv \boldsymbol{Q}_p$.

The following analogue of our main result can be proved using much the same techniques.

THEOREM 4.2. *Suppose* $\lambda \geqq \kappa > \aleph_0$ *are cardinals with* κ *regular. Then the following are equivalent*:

(1) *There is a* BW(κ) *p-adically closed field of cardinality* λ.

(2) *There is a* κ*-tree with exactly* λ *branches and with no* κ*-Aronszajn subtrees.*

PROOF. (Sketch) (1) $\Rightarrow$ (2). The object is to construct a tree in a manner similar to the way Keisler trees were constructed in §1. Let K be the given field and G its value group. Then G has cofinality κ, so let $\langle c_\nu : \nu < \kappa \rangle$ be an increasing, cofinal sequence of positive elements of G. For each $\nu < \kappa$ let $\sim_\nu$ be an equivalence relation defined on K as follows: $x \sim_\nu y$ iff $v(x-y) \geqq c_\nu$. Let T be the set of all equivalence classes of all the equivalence relations $\sim_\nu$. Then $(T, \supseteq)$ is a κ-tree having exactly λ branches with no κ-Aronszajn subtrees.

(2) $\Rightarrow$ (1). As in §3 let $\mathcal{N} \models \mathrm{PA}$ be compact, have cardinality κ, and have exactly λ branches. From $\mathcal{N}$ construct $\boldsymbol{Q}_p^{\mathcal{N}}$, the p-adic numbers of $\mathcal{N}$. This can be done, as in §2, by considering all formal infinite series $\Sigma_{i \in N} a_i p^i$, where $a_i < p$ and $\{\langle i, a_i \rangle : i \in N\} \in \mathrm{Class}(\mathcal{N})$. In the usual way, one can show that $\boldsymbol{Q}_p^{\mathcal{N}}$ satisfies Hensel's Lemma (so by Theorem 4.1, $\boldsymbol{Q}_p^{\mathcal{N}} \equiv \boldsymbol{Q}_p$). Obviously, $|\boldsymbol{Q}_p^{\mathcal{N}}| = \lambda$. It is easy to see that $\boldsymbol{Q}_p^{\mathcal{N}}$ is $\mathrm{BW}(\kappa)$. ■

There are also corresponding analogues of Corollaries 3.4 and 3.5.

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